These figures provide a dramatic illustration of the improvement of accuracy provided by the use of the consistent mass matric approach to such problems. Even with a 7 by 7 grid the lumped mass matrix approach was still 3.2% different from Lamb's value of $\omega_1{}^2$.

In conclusion, it should be pointed out that the application of the fluid incompressibility constraint conditions renders the reduced form of the mass and stiffness matrices fully populated in both cases and consequently the arguments which favoured the lumped mass matrix approach for structural problems disappear and from the above table it is clear that only a consistent mass matrix should be used.

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Stability Derivatives for Bodies of Revolution at Subsonic Speeds

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I. Introduction

RECENT interest in slender wing/body and airship design in subsonic flight demands improved techniques for their static and dynamic stability prediction. For the steady lifting case, Refs.1-4 have derived higher approximations based on the linearized small-perturbation potential equation, i.e., Eq. (4). More exact theories were given by Hess and Smith[5] in a numerical approach and by Revell[6] in a matched asymptotic expansion procedure. For the oscillatory case, Revell's work appears to be the only available one, suitable for the dynamic-stability calculation.

The purpose of this Note is twofold. First, the present method employs a comparatively simpler procedure than Revell's in extending the earlier work[1-4] to the unsteady case. The result obtained can be expressed simply in algebraic form, and therefore it can be adopted easily for practical design calculations. Second, the present investigation serves as a numerical check on the limitation of the subsonic linearized equation in comparison to the second-order equation derived by Revell.

In Revell's formulation, the oscillatory-flow equations [Eqs. (27) and (54) of Ref. 6] are both inhomogeneous equations that account for the coupling effects with the steady mean flow. In the present formulation, the governing equation used is a linear homogeneous one, in which the mean flow influence is absent.[§] Nevertheless, the mean flow influence is recovered partly in the present solution by asymptotic expansion of the near-flowfield through the flow tangency condition, Eq. (11), and the pressure formula Eqs. (14) and (15). In this way, the effects of freestream Mach number M, body shape $R(x)$, and body thickness ϵ can be accounted for in the stability derivative calculation. We remark that the present procedure is essentially the same as the quasislender body theory for supersonic flow given by Platzer and Hoffman[7] and is considered to be its subsonic counterpart.

II. Problem Formulation

Consider a rigid, pointed body of revolution which is exposed to a steady uniform subsonic flow and which performs harmonic, small-amplitude pitching oscillations around its zero angle of attack position. A body-fixed cylindrical coordinate system, as shown in Fig. 1, is used to describe the problem.[¶] The body is assumed to be smooth and sufficiently slender so that the small-perturbation concept can be applied. Let δ_0 represent the amplitude of oscillation and k the reduced frequency. Following Revell,[8] a body-fixed perturbation potential $\Phi(x,r,\theta,t)$ can be related to the general velocity potential $\Omega(x,r,\theta,t)$ as follows:

$$\Omega(x,r,\theta,t)=(x-a)\ \cos\delta+r\ \sin\delta\ \cos\theta+\Phi(x,r,\theta,t) \qquad (1)$$

where

$$\delta=\delta_0\ e^{ikt} \qquad (2)$$

is the pitch angle. The velocity components in the x,r,θ directions then are given by

$$u=\Omega_x=\cos\delta\ +\ \Phi_x \qquad (3a)$$

$$v=\Omega_r=\sin\delta\ \cos\theta\ +\ \Phi_r \qquad (3b)$$

$$w=(1/r)\Omega_\theta=-\sin\delta\ \sin\theta\ +\ (1/r)\Phi_\theta \qquad (3c)$$

The first-order equation governing unsteady subsonic flow is

$$(1-M^2)\Phi_{xx}\ +\ \Phi_{rr}\ +\ (1/r)\Phi_r\ +\ (1/r^2)\Phi_{\theta\theta}$$
$$-2M^2\Phi_{xt}-M^2\Phi_{tt}=0 \qquad (4)$$

For a discussion of the parametric restrictions imposed on this equation, see Ref. 10. Assuming harmonic time dependence, the perturbation potential can be written as

$$\Phi(x,r,\theta,t)=\phi(x,r)\ +\ \varphi(x,r,\theta)e^{ikt} \qquad (5)$$

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§The authors are grateful to J. D. Revell for stressing this point.

¶All variables are nondimensional, with distances referred to body length, velocities to freestream speed, and time to body length divided by freestream speed.

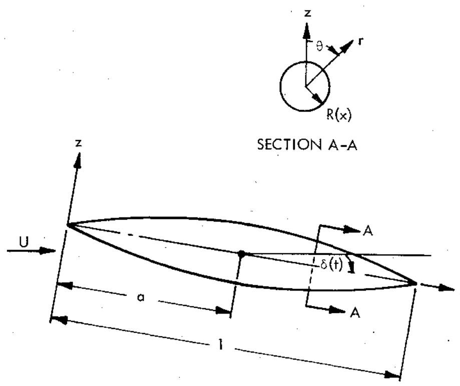

Fig. 1 Body-fixed coordinate system.

where $\phi(x,r)$ describes the zero angle of attack and $\varphi(x,r,\theta)$ the oscillatory flow problem. Substituting Eq. (5) into Eq. (4) gives

$$(1-M^2)\phi_{xx} + \phi_{rr} + (1/r)\phi_r = 0 \tag{6a}$$

$$(1-M^2)\varphi_{xx} + \varphi_{rr} + (1/r)\varphi_r + (1/r^2)\varphi_{\theta\theta} - 2\mathrm{i}kM^2\varphi_x + k^2M^2\varphi = 0 \tag{6b}$$

There are several ways to obtain appropriate asymptotic solutions of these equations, and we refer to Ref. 9 for a comprehensive outline of these approaches. Here, only the main results are indicated. Fourier transforming Eq. (6b) in the x direction, one obtains

$$\bar\varphi_{rr} + (1/r)\bar\varphi_r + (1/r^2)\bar\varphi_{\theta\theta} - \lambda^2\bar\varphi = 0 \tag{7}$$

$$\lambda^2 = B^2 + u^2 + 2kM^2u - k^2M^2, \quad \beta^2 = 1 - M^2 \tag{7a}$$

where

$$\bar\varphi(u,r,\theta) = \int_{-\infty}^{+\infty} e^{\mathrm{i}ux}\varphi(x,r,\theta)\,\mathrm{d}x \tag{7b}$$

Application of Sommerfeld's radiation condition then leads to the dipole solution appropriate for oscillating (lifting) bodies of revolution:

$$\bar\varphi(u,r,\theta) = \lambda\bar F(u)K_1(\lambda r)\cos\theta$$

where $\bar F(u)$ is the transformed doublet strength. Expanding the modified Bessel function $K_1(\lambda r)$ for small arguments and retaining the first two terms of the series leads, after inversion, to the following solution:

$$\varphi(x,r,\theta) = \frac{F(x)}{2\pi r}\cos\theta + \frac{r\cos\theta}{8\pi} \times \Lambda\left[F(x)\left(\ell n\frac{\beta^2r^2}{4} - 1\right)\right] - \frac{r\cos\theta}{8\pi}\Lambda\left\{\left[\frac{\partial}{\partial x} + \frac{\mathrm{i}kM}{1+M}\right] \times \int_0^x F(\xi)\exp\left[-i\frac{kM(x-\xi)}{1+M}\right]\ell n(x-\xi)\,\mathrm{d}\xi - \left[\frac{\partial}{\partial x} - \frac{\mathrm{i}kM}{1-M}\right]\int_x^1 F(\xi) \exp\left[-i\frac{kM(\xi-x)}{1-M}\right]\ell n(\xi-x)\,\mathrm{d}\xi \right. \tag{8}$$

where

$$\Lambda = -\beta^2(\partial^2/\partial x^2) + 2\mathrm{i}kM^2(\partial/\partial x) - k^2M^2$$

For stability and control work, it usually is sufficient to account only for low-frequency effects. Hence, Eq. (8) can be simplified by retaining only terms up to the first power in frequency, i.e.,

$$\varphi(x,r,\theta) = \{[F(x)/2\pi r] + A(x,r,\beta)\}\cos\theta \tag{9}$$

where

$$A(x,r,\beta) = -\frac{\beta^2 r}{8\pi}\left\{\left[\ell n\frac{\beta^2r^2}{4} - 1\right]F''(x) - \frac{\partial^3}{\partial x^3}\int_0^x F(\xi)\ell n(x-\xi)\,\mathrm{d}\xi + \frac{\partial^3}{\partial x^3}\int_x^1 F(\xi)\ell n(\xi-x)\,\mathrm{d}\xi\right\} + i\frac{kM^2r}{4\pi}\left\{\ell n\frac{\beta^2r^2}{4}F'(x) - \frac{\partial^2}{\partial x^2}\int_0^x F(\xi)\ell n(x-\xi)\,\mathrm{d}\xi + \frac{\partial^2}{\partial x^2}\int_x^1 F(\xi)\ell n(\xi-x)\,\mathrm{d}\xi\right\} \tag{9a}$$

Other equivalent solutions, but somewhat different in their final expressions, have been obtained in Ref. 9 by extending the work of Laitone,[1] Schultz-Piszachich,[2] and Keune[4] for steady flow past bodies of revolution at angle of attack to slowly oscillating bodies. It is straightforward to show their equivalence, and we merely list here the extended Laitone solution because of its advantages from a programming standpoint (in particular, for power-law bodies):

$$\begin{aligned} A(x,r,\beta) = &-\frac{\beta^2r}{4\pi}\left\{\frac{F(x)}{2}\left[\frac{1}{(1-x)^2} + \frac{1}{x^2}\right] + F'(x) \times \left[\frac{1}{1-x} - \frac{1}{x}\right] + F''(x)\left[1 + \ln\frac{\beta r}{2[x(1-x)]^{1/2}}\right] - \sum_{n=3}^{\infty}\frac{F^{(n)}(x)}{n!(n-2)}\left[(1-x)^{n-2} + (-x)^{n-2}\right]\right\} \\ &- \frac{ikM^2r}{4\pi}\left\{F(x)\left[\frac{1-2x}{x(1-x)}\right] - 2F'(x)\left[\ln\frac{\beta r}{2[x(1-x)]^{1/2}} + 1\right] + \sum_{n=2}^{\infty}\frac{F^{(n)}(x)}{n!(n-1)}\left[(1-x)^{n-1} + (-x)^{n-1}\right]\right\} \end{aligned} \tag{10}$$

The boundary condition at the body requires that the flow be tangent to the body surface at every instant of time. This

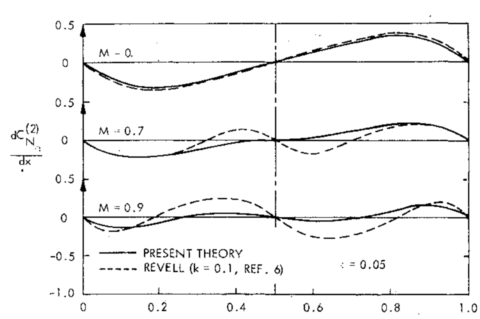

Fig. 2 Second-order normal force distributions for a parabolic spindle of thickness ratio $\epsilon = 0.05$.

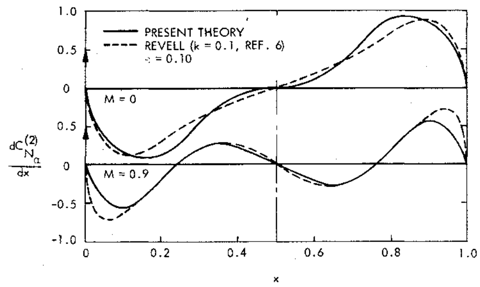

Fig. 3 Second-order normal force distribution for a parabolic spindle of thickness ratio $\epsilon = 0.10$.

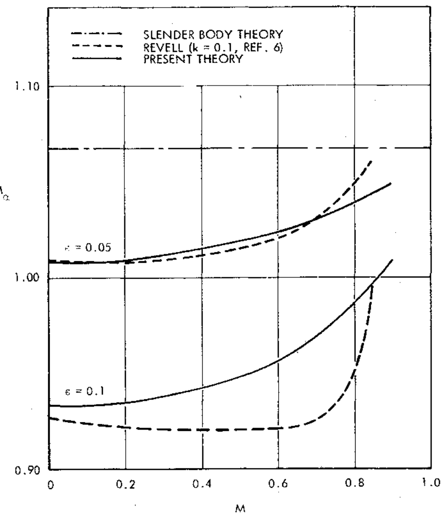

Fig. 4 Effect of Mach number and thickness ratio on pitching moment coefficient slope for a parabolic spindle ($a = 0$).

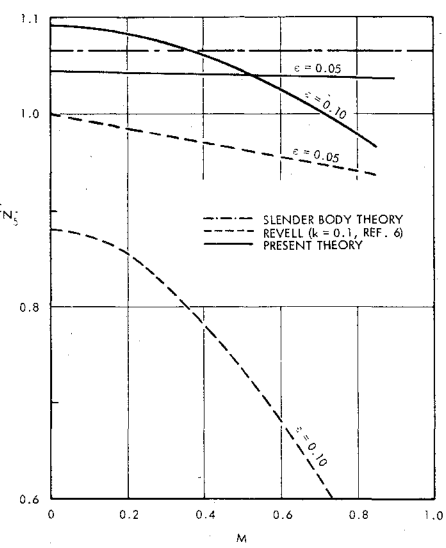

Fig. 5 Effect of Mach number and thickness ratio on damping-in-pitch normal force coefficient slope for a parabolic spindle ($a = 0$).

can be expressed as follows:

$$\phi_r = R'(x)\ (1 + \phi_x) \tag{11a}$$

$$\psi_r = -1 + R'(x)\ \psi_x \tag{11b}$$

$$\lambda_r = -[(x-a) + R(x)\ R'(x)] + R'(x)\ \lambda_x \tag{11c}$$

where the perturbation potential $\varphi(x,r,\theta)e^{ikt}$ was split into in-

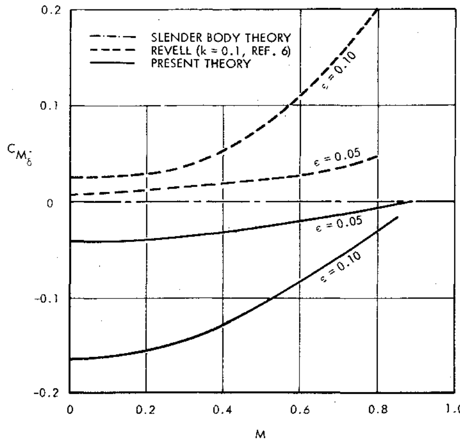

Fig. 6 Effect of Mach number and thickness ratio on damping-in-pitch moment coefficient slope for a parabolic spindle ($a = 0$).

phase and out-of-phase potentials ψ and λ:

$$\varphi(x,r,\theta,k)e^{ikt}/\cos\theta = \delta\ \psi(x,r) + \dot\delta\ \lambda(s,r) = \chi(x,r) \tag{12}$$

in accordance with the low-frequency assumption. Using the Adams-Sears iteration method[3] to determine the doublet distribution $F(\xi)$ in Eq. (9), this potential is expanded into

$$\chi(x,r) = \chi^{(1)}(x,r) + \chi^{(2)}(x,r) + \ldots \tag{13}$$

where the first term is the slender body term (of order ϵ), and the second term represents a correction term due to body thickness and Mach number effects (of order $\beta^2\epsilon^3, \beta^2\epsilon^3 \ln\beta\epsilon$). By similar expansion of the doublet distribution, the in-phase and out-of-phase potentials can then be determined and are listed in Ref. 11.

Using Bernoulli's equation, the in-phase and out-of-phase pressures are obtained to second order as

$$C_{p_1}(x,R) = -2\{\psi_x^{(1)+(2)}(x) + \beta^2\psi_x^{(1)}(x)\phi_x^{(1)}(x) + [1-(M^2/2)]R'^2(x)\psi_x^{(1)}(x)\} \tag{14}$$

$$C_{p_2}(x,R) = -2\{(\lambda_x+\psi)^{(1)+(2)} - [-\beta^2\lambda_x^{(1)}(x) + M^2\psi^{(1)}(x) + R(x)]\phi_x^{(1)}(x) + R'^2(x)\{[1-(M^2/2)]\lambda_x^{(1)}(x) - (M^2/2)\psi^{(1)}(x) - R(x)]\} \tag{15}$$

The stability derivatives C_{N_α}, C_{m_α}, $C_{N_{\dot\delta}}$, and $C_{m_{\dot\delta}}$ are identical to those defined in Ref. 7 [Eqs. (43-49), pp. 15-16].

III. Numerical Results and Discussion

Sample calculations are presented for the normal force distributions as well as total force and moment coefficients for parabolic spindles $R(x) = 4\epsilon(1-x)x$. In each case, a comparison is made with Revell's second-order slender body theory.[6] His theory provides corrections for thickness and compressibility effects by deriving successive approximations to the complete second-order potential equation for nonviscous time-dependent subsonic flow. Therefore, Revell's solution is taken as the proper reference solution for an evaluation of the present theory.

Figures 2 and 3 show a comparison of the corrections to slender body theory for the normal force distributions over parabolic arc bodies at various Mach numbers. The present theory and Revell's theory virtually coincide for bodies of thickness ratio $\epsilon = 0.05$ and are still in quite good agreement for $\epsilon = 0.1$. This is demonstrated further in Fig. 4 for the pitching moment coefficient $C_{M\alpha}$ at the high subsonic Mach numbers being predicted by Revell for the $\epsilon = 0.1$ body.

In Figs. 5 and 6, a comparison is given between the present theory and Revell's computations for the effect of Mach number and thickness ratio on the damping-in-pitch normal force and moment coefficient slopes of a parabolic spindle. Although the two theories predict similar trends with Mach number, there is a difference in absolute levels in each case which may be caused by the difference in coupling with the mean flow and/or by Revell's inclusion of the quadratic frequency terms.

IV. Summary

A theory was developed for subsonic flow past slowly oscillating pointed bodies of revolution. By properly expanding the first-order velocity potential, correction terms to slender-body theory could be derived which account for thickness and compressibility effects. This work is an extension of previous approaches by Adams and Sears[3], Laitone,[1] and Schultz-Piszachich[2] for flow past bodies at steady angle of attack, and the equivalence of these approaches is pointed out. The numerical results show good agreement between Revell's second-order slender-body theory and the present theory for the static stability derivatives of parabolic spindles. Thus it gives confidence in the use of this much simpler theory for slender bodies. Further experiments are required to assess the reliability of the dynamic stability predictions of both theories.

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Governing Equations for Large Deflections of Sandwich Plates

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Nomenclature

a,b	= linear dimensions of rectangular sandwich plate
E	= Young's modulus of face
$F(b/a)$	= correction function, Eq. (27)
G'	= shear modulus of core
h	= core thickness (measured from middle surfaces of each face)
I_1^m, I_2^m	= first and second invariants of averaged strains
q	= lateral load
t	= face thickness
u,v,w	= displacements in x, y, and z directions, respectively
$\bar{V}_0$	= strain energy per unit area
x,y,z	= rectangular coordinate system
ϵ,γ	= strain components
σ,τ	= stress components
ν	= Poisson's ratio of face
$(\ )^c$	= core variable
$(\ )^f$	= face variable
$(\ )^u$	= upper face variable
$(\ )^\ell$	= lower face variable
$(\ )^m$	= averaged value

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Introduction

MANY investigations have been carried out on finite deformations of sandwich plates and shells, which have practical importance.[1-6] About two decades ago, Berger[7] proposed an interesting approximate method for large deflection problems of single-layer isotropic plates. His method is based on unqualified disregard of a second invariant of middle plane strains of plate (membrane strains) in the strain energy formula, and then by the variational calculus linear system of equations were derived with respect to displacements, in which the deflection is decoupled with inplane displacements. This idea was extended to static and dynamic problems of anisotropic as well as isotropic plates and shallow shells.[8-10]

However, so far as we know, no application of it to sandwich plates and shells has been reported. Since deformation responses of faces and core of sandwich structures are different, it is only natural that Berger's original idea may not be directly applicable. In this Note, we deal with an average of fiber strains of each face, which is algebraically identical to the membrance strain of conventional plates, and then we apply the foregoing Berger's method to a symmetrically-laminated sandwich plate with large deflection. Numerical illustrations of laterally loaded rectangular sandwich plates are presented.

Governing Equations

First, we posit a rectangular coordinate system x, y, z: x, y in the middle plane of the core, z thickness direction (positive downward). For the sake of simplicity, consider a sandwich plate with an isotropic core as well as isotropic upper and lower faces of identical thickness. While the faces respond to the bending and membrance actions of the plate, the core is assumed to transfer only shear deformations. Moreover, compared with the core thickness h, the face thickness t is supposedly thin enough to ignore a variation of stress in the thickness direction of the faces.

The strain components of each face are expressed, in terms of each displacement components, as

$$\epsilon_x^u = \frac{\partial u^u}{\partial x} + \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \epsilon_y^u = \frac{\partial v^u}{\partial y} + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,$$

$$\gamma_{xy}^u = \frac{\partial u^u}{\partial y} + \frac{\partial v^u}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial x}, \tag{1a}$$

$$\epsilon_x^\ell = \frac{\partial u^\ell}{\partial x} + \tfrac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \epsilon_y^\ell = \frac{\partial v^\ell}{\partial y} + \tfrac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2,$$

$$\gamma_{xy}^\ell = \frac{\partial u^\ell}{\partial y} + \frac{\partial v^\ell}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial w}{\partial x} \tag{1b}$$